On the Characteristic Curvature Operator

Vittorio Martino[1]

Abstract We introduce the Characteristic Curvature as the curvature of the trajectories of the Hamiltonian vector field with respect to the normal direction to the isoenergetic surfaces; by using the Second Fundamental Form we relate it to the Classical and Levi Mean Curvature. Then we prove existence and uniqueness of viscosity solutions for the related Dirichlet problem and we show the Lipschitz regularity of the solutions under suitable hypotheses. At the end we show that neither Strong Comparison Principle nor Hopf Lemma hold for the Characteristic Curvature Operator.

1 Introduction

In this paper we introduce the Characteristic Curvature as the curvature of the trajectories of the Hamiltonian vector field with respect to the normal direction to the isoenergetic surfaces. Namely, let H be a smooth Hamiltonian function on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ equipped with its standard symplectic structure J; then the level set M of H, corresponding to some noncritical energy value E, is a smooth hypersurface in $\mathbb{R}^{2n+2}$. The Hamiltonian vector field X^H is the tangent vector field to M, defined by $X^H := JDH$. The orbits of X^H are the critical points of the Action functional defined on a suitable space of curves; therefore they represent the trajectories of the motion in the generalized phase space. In particular they are curves on M: we will define the characteristic curvature $\mathcal{C}^M$ as the normalized curvature of these curves with respect to the unit normal direction to M; we will say that M is strictly $\mathcal{C}$-convex if $\mathcal{C}^M > 0$. Later, since M is a real hypersurface in $\mathbb{C}^{n+1}$, by using the Second Fundamental Form and the Levi Form we relate $\mathcal{C}^M$ to the Classical Mean Curvature $\mathcal{H}^M$ and to the Levi Mean Curvature $\mathcal{L}^M$. We want explicitly to note that the characteristic curvature $\mathcal{C}^M$ can be used to obtain characterization properties: in fact, following some results

[1]Department of Mathematics, Rutgers University - Hill Center for the Mathematical Sciences 110 Frelinghuysen Rd., Piscataway 08854-8019 NJ, USA. E-mail address: martino@dm.unibo.it

obtained by Hounie and Lanconelli ([7], [8]) in which they prove Alexandrov type theorems for Reinhardt domains by using the Levi Mean Curvature, it is proved in [11] an analogous symmetry result for Reinhardt domains, starting from the characteristic curvature $\mathcal{C}^M$.
Denoting by $\mathcal{T}$ the related differential operator, we will find the explicit expression $\mathcal{T}u = tra\big(\widetilde{A}(Du)\, D^2u\big)$, being $\widetilde{A}$ a symmetric matrix defined in the sequel. The characteristic curvature operator $\mathcal{T}$ is a quasilinear (highly) degenerate elliptic operator on $\mathbb{R}^{2n+1}$: in fact the principal part has $2n$ distinct eigenvectors corresponding to the eigenvalue zero and only one direction of positivity. Let Ω be a bounded open set in $\mathbb{R}^{2n+1}$, then under suitable hypotheses we will prove existence and uniqueness of viscosity solutions for the associated Dirichlet Problem, with prescribed curvature function $k \in C(\Omega \times \mathbb{R})$:

$$\begin{cases} F(x,u,Du,D^2u) := -tra(\widetilde{A}(Du)D^2u) + k(x,u) = 0 & \text{in } \Omega, \\ u(x) = \varphi(x), & \text{on } \partial\Omega, \end{cases} \qquad (DP)$$

where $\varphi \in C(\partial\Omega)$. In order to do that we will use the classical tools introduced by Crandall, Ishii, Lions in [3], [9] and we will give geometric sufficient conditions on Ω and on the prescribed curvature k in order to ensure the existence of sub- and supersolutions for (DP). Namely, if we denote by $\Omega_c := \partial\Omega \times \mathbb{R}$, the cylinder type hypersurface in $\mathbb{R}^{2n+2}$, then we will assume:
let Ω_c be a strictly $\mathcal{C}$-convex hypersurface, then

$$\sup_{s\in\mathbb{R}} k(x,s) < \mathcal{C}_x^{\Omega_c}, \quad \textit{for every } x \in \partial\Omega; \qquad (1)$$

and
let R be the radius of the smallest ball containing Ω, then

$$\sup_{\overline{\Omega}\times\mathbb{R}} k \le \frac{1}{R} \qquad (2)$$

We will prove the following result:

Theorem 1.1. *Let $\partial\Omega \in C^2$ and suppose (1) and (2) hold. If k is either strictly increasing with respect to u or non-decreasing with respect to u but independent of x, then there exists a unique viscosity solution for (DP).*

Later we will prove the Lipschitz regularity of solutions: first we use a Bernstein method to obtain a gradient bound for the solutions of the regularized equation and then we use a limit process argument. In particular we need

a slightly stronger assumption than (1):
let Ω_c be a strictly $\mathcal{C}$-convex hypersurface such that there exists a defining function for Ω, $\rho \in C^{2,\alpha}$, $0 < \alpha < 1$, with $\triangle\rho > 0$ on $\partial\Omega$, then

$$\sup_{s\in\mathbb{R}} k(x,s) < \mathcal{C}_x^{\Omega_c} \quad \textit{for every } x \in \partial\Omega. \tag{3}$$

Remark 1.2. *The hypothesis of having a defining function with $\triangle\rho > 0$ is obviously fulfilled if $\partial\Omega$ is strictly convex; it is also satisfied if the cylinder Ω_c is strictly pseudoconvex as hypersurface in $\mathbb{C}^{n+1}$.*

Therefore we prove:

Theorem 1.3. *Let us suppose that the hypotheses (2) and (3) hold. Let $k \in C^1(\overline{\Omega}\times\mathbb{R})$ and $\varphi \in C^{2,\alpha}(\partial\Omega)$, $0 < \alpha < 1$. If*

$$i)\frac{\partial k}{\partial u} \geq 0 \tag{4}$$

$$ii)k^2 - \sum_{k=1}^{2n+1}\left|\frac{\partial k}{\partial x_k}\right| \geq 0 \tag{5}$$

then (DP) has a Lipschitz continuous viscosity solution. Moreover, if k is either strictly increasing with respect to u or non-decreasing with respect to u but independent of x then the solution is unique.

We then show a non-existence result on balls when the prescribed curvature is a positive constant. Similar results were proved by Slodkowski and Tomassini in [13] for the Levi equation in the case $n = 1$; by Martino and Montanari in [12] for the Mean Levi Curvature; by Slodkowski and Tomassini in [14] and by Da Lio and Montanari in [4] for the Levi Monge Ampère equation.
At the end, by mean of two counterexamples, we will show that neither the Strong Comparison Principle nor the Hopf Lemma hold for the operator $\mathcal{T}$. This is substantial difference between the highly degenerate Characteristic operator and the Levi Curvature operators, for which Lanconelli and Montanari in [10] proved the Strong Comparison Principle: indeed the principal part of Levi Curvature operators is degenerate only with respect to one direction and when computed on strictly pseudoconvex functions, the $2n$ vector fields, respect to which the operator is strictly elliptic, satisfy the Hörmander rank condition.

Acknowledgement I wish to thank Professor Annamaria Montanari and Professor YanYan Li for their useful suggestions and comments on the preparation of this work.
Moreover this paper was completed during the year that I spent at the Mathematics Department of Rutgers University: I want to express my gratitude for the hospitality and I'm grateful to the Nonlinear Analysis Center for its support.

2 The characteristic curvature

Here we recall some known facts and we refer for instance to [6] for a full detailed exposition. Let us consider $z = (x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. We will denote by $\lambda = (1/2)\sum_{k=1}^{n+1}(y_k dx_k - x_k dy_k)$ the standard Liouville differential 1-form and by $\omega := d\lambda$ the canonical symplectic 2-form. We will also denote by g the standard scalar product in $\mathbb{R}^{2n+2}$, and by J the canonical symplectic matrix in $\mathbb{R}^{2n+2}$. Let us consider a smooth Hamiltonian function $H : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}, \quad z = (x, y) \longmapsto H(x, y) = H(z)$, and let M be the isoenergetic hypersurface in $\mathbb{R}^{2n+2}$ defined by $M = \{z \in \mathbb{R}^{2n+2} : H(z) = E\}$, with $DH(z) \neq 0$ for all $z \in M$, where E is some constant. The trajectories of motion are solutions of the following first order system (Hamilton)

$$\dot{x}_k = \frac{\partial H}{\partial y_k}(x, y), \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}(x, y), \qquad k = 1, \ldots, n+1 \tag{6}$$

Moreover if γ solves (6), then $\gamma \subseteq M$. Now we introduce the Hamiltonian vector field $X_z^H := JDH(z)$; then the Hamilton system (6) rewrites as $\dot{\gamma} = X_\gamma^H$. We explicitly remark that the direction given by the Hamiltonian vector field only depends on M and J. By taking the restriction of ω on TM, one has

$$rank(\omega|_{TM}) = 2n \qquad \text{and} \qquad dim(ker(\omega|_{TM})) = 1$$

We introduce then the following one-dimensional subspace of the tangent space:

$$K_z = \{\xi \in T_z M : \ \omega(v, \xi) = 0, \ \forall v \in T_z M\}$$

A smooth curve $\gamma \subseteq M$, such that $\dot{\gamma} \in K_\gamma$ is called a characteristic curve on M. Since

$$\omega(v, X^H) = \omega(v, JDH) = g(v, DH) = 0, \quad \forall v \in TM,$$

therefore $X_z^H \in K_z$, $\forall z \in M$, and its orbits are characteristic curves on M.

Remark 2.1. *Since $ker(\omega|_{TM})$ is a one-dimensional subspace and X^H (which never vanishes) always belongs to $ker(\omega|_{TM})$, then a smooth curve $\gamma \subseteq M$ is characteristic, up to reparametrization, if and only if $\dot{\gamma} = X^H$.*

We want to compute the curvature of the characteristic curves with respect to the normal direction to M.

Definition 2.2. *Let $\varepsilon > 0$ and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be any smooth curve such that $\gamma(0) = z \in M$ and $\dot{\gamma}(0) \in K_{\gamma(0)}$. We will call the characteristic curvature of M at z the following*

$$\mathcal{C}_z^M := \frac{g(\ddot{\gamma}(0), N_z)}{|\dot{\gamma}(0)|^2}$$

where N_z is a unit normal direction to M at z. We will say that M is strictly $\mathcal{C}$-convex if $\mathcal{C}_z^M > 0$, for every $z \in M$.

We can obtain a formula for the characteristic curvature explicitly involving only the characteristic curves. In fact, let $\gamma \subseteq M$ be a characteristic curve, then a unit normal direction along γ is given by $N_\gamma = J\dot{\gamma}/|\dot{\gamma}|$, and therefore

$$\mathcal{C}_\gamma^M := \frac{g(\ddot{\gamma}, J\dot{\gamma})}{|\dot{\gamma}|^3}$$

Remark 2.3. *By the previous formula we can see that the characteristic curvature is a scalar invariant under (rigid) symplectic diffeomorphisms.*

We will add some explicit examples at the end of the paper.

3 Relation with the Classical and Levi Mean Curvature

Let M be a smooth real hypersurface in $\mathbb{C}^{n+1}$ and let us identify $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$, with $z = (z_1, \ldots, z_{n+1}), z_k = x + iy \simeq (x_k, y_k)$. A defining function for M is a function $f : \mathbb{C}^{n+1} \to \mathbb{R}$ such that

$$\Omega = \{z \in \mathbb{C}^{n+1} : f(z) < 0\}, \quad M = \partial\Omega = \{z \in \mathbb{C}^{n+1} : f(z) = 0\},$$

and $Df \neq 0$ on $\partial\Omega$. Let $N = -Df/|Df|$ be the (inner, if M is compact) unit normal, we define the characteristic direction $T \in TM$ as $T := -J(N)$. Therefore the characteristic direction for M is the normalized Hamiltonian vector field. The complex maximal distribution or Levi distribution HM is

the largest subspace in TM invariant under the action of J, namely $HM = TM \cap J(TM)$. Moreover TM splits into a direct sum, $TM = HM \oplus_g \mathbb{R}T$ where $dim(HM) = 2n$ and the sum is g-orthogonal. Let us denote by ∇ the Levi-Civita connection in $\mathbb{C}^{n+1}$ and by h the Second Fundamental Form on TM. The Levi Form l is the hermitian operator on HM defined in the following way: $\forall X_1, X_2 \in HM$, if $Z_1 = X_1 - iJ(X_1)$ and $Z_2 = X_2 - iJ(X_2)$, then

$$l(Z_1, \bar{Z}_2) = g(\nabla_{Z_1} \bar{Z}_2, N) \tag{7}$$

We can compare the Levi Form with the Second Fundamental Form (see [2], Chap.10, Theorem 2):

$$\forall X \in HM, \quad l(Z, \bar{Z}) = h(X, X) + h(J(X), J(X)) \tag{8}$$

Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$, with $Y_k = JX_k$, be an orthonormal basis of the horizontal space HM; then the Second Fundamental Form has the following structure

$$h = \begin{pmatrix} h(X_k, X_k) & h(X_k, Y_j) & h(X_k, T) \\ h(Y_j, X_k) & h(Y_j, Y_j) & h(Y_j, T) \\ h(T, X_k) & h(T, Y_k) & h(T, T) \end{pmatrix}$$

with $k, j = 1, \ldots, n$. Moreover, by the very definition of characteristic curvature we have $h_z(T, T) = g(\nabla_T T, N) = \mathcal{C}^M_z$, for every $z \in M$.

Remark 3.1. *In this setting we see that the characteristic curvature depends only on M and on the complex structure J, therefore it is a scalar invariant under (rigid) holomorphic diffeomorphisms.*

The classical Mean Curvature H^M and the Levi Mean Curvature L^M are respectively:

$$\mathcal{H}^M = \frac{1}{2n+1} tra(h), \quad \mathcal{L}^M = \frac{1}{n} tra(l) \tag{9}$$

Therefore a direct computation leads to the relation between $\mathcal{H}^M$, $\mathcal{L}^M$ and $\mathcal{C}^M$:

$$(2n+1)\mathcal{H}^M = (2n\mathcal{L}^M + \mathcal{C}^M) \tag{10}$$

4 The operator

Here we find an explicit formula for $\mathcal{C}^M$ that involves a defining function f. A direct computation shows that for any $z \in M$ we have:

$$\mathcal{C}^M_z := \frac{1}{|Df(z)|^3} g(D^2 f(z) JDf(z), JDf(z)) = \frac{1}{|Df(z)|^3} tra\Big(A(Df(z)) D^2 f(z)\Big)$$

where A is the following $(2n+2)\times(2n+2)$ symmetric matrix:

$$A(Df(z)) = \begin{pmatrix} f_y \otimes f_y & -f_y \otimes f_x \\ -f_x \otimes f_y & f_x \otimes f_x \end{pmatrix}$$

We define the characteristic curvature operator $\mathcal{T}$ as the differential second order operator acting on f in the following way:

$$\mathcal{T}f(z) := \frac{1}{|Df(z)|^3} tra\Big(A(Df(z))D^2 f(z)\Big)$$

We are interested in finding an expression for $\mathcal{T}$ when we locally consider the hypersurface M as the graph of some function $u : \mathbb{R}^{2n+1} \supseteq \Omega \to \mathbb{R}$ such that $(\xi, u(\xi)) \in M$ for all $\xi \in \Omega$. In order to do that, we set

$$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n), \quad x_{n+1} = t, \quad y_{n+1} = s, \quad \xi = (x, y, t)$$

and we take as defining function

$$f(z) = f(x, y, t, s) = u(x, y, t) - s = u(\xi) - s, \qquad |Df|^2 = 1 + |Du|^2$$

Then we have

$$\mathcal{T}u := \frac{1}{(1+|Du|^2)^{\frac{3}{2}}} tra\Big(A(Du)\, D^2 u\Big)$$

where A is the following symmetric matrix:

$$A(Du) = \begin{pmatrix} u_y \otimes u_y & -u_y \otimes u_x & -u_y \\ -u_x \otimes u_y & u_x \otimes u_x & u_x \\ -u_y & u_x & 1 \end{pmatrix} \tag{11}$$

Example 4.1 (n=1)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be an open set, and $u : \Omega \to \mathbb{R}$ a C^2 function. Then*

$$\mathcal{T}u = \frac{1}{(1+|Du|^2)^{\frac{3}{2}}}\Big(u_y^2 u_{xx} + u_x^2 u_{yy} + u_{tt} - 2u_x u_y u_{xy} + 2u_x u_{yt} - 2u_y u_{xt}\Big)$$

The characteristic curvature operator $\mathcal{T}$ is a second order quasilinear (highly) degenerate elliptic operator on $\mathbb{R}^{2n+1}$: in fact, by (11) we see that the following $2n$ independent vector fields

$$\partial_{x_k} + u_{y_k}\partial_t \quad , \qquad \partial_{y_k} - u_{x_k}\partial_t \quad , \qquad k = 1, \dots, n$$

are eigenvectors for A with eigenvalue identically equals to zero; instead the vector field

$$-u_{y_1}\partial_{x_1} - u_{y_n}\partial_{x_n} + u_{x_1}\partial_{y_1} + u_{x_n}\partial_{y_n} + \partial_t$$

is an eigenvector for A with eigenvalue equals to $(1+|u_x|^2+|u_y|^2)$. For the sake of simplicity we will call $\widetilde{A}(p) = \frac{1}{(1+|p|^2)^{\frac{3}{2}}}A(p),\ \forall p \in \mathbb{R}^{2n+1}$, so that $\mathcal{T}u = tra\big(\widetilde{A}(Du)\, D^2u\big)$. Moreover we can write $\widetilde{A}(p) = \widetilde{\sigma}(p)\widetilde{\sigma}(p)^T$, where

$$\widetilde{\sigma}(Du) = \frac{1}{(1+|p|^2)^{\frac{3}{4}}} \begin{pmatrix} -u_y \\ u_x \\ 1 \end{pmatrix}$$

5 Viscosity solutions

Here we prove Theorem (1.1). We refer the reader to [3], [9] for a complete exposition regarding the theory of viscosity solutions. If k is a prescribed continuous function, non-negative and strictly increasing with respect to u, then F is proper according the definition in [3] and then the comparison principle for F holds. Anyway, since we are interested even at the case when the characteristic curvature is constant, we would like to have a comparison principle for F also when k is not strictly increasing with respect to u, but it does not depend on x. We will adapt the proof for the strictly monotone case: in order to do that we need two standard lemmas and we refer the reader to [3] for the proofs.

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^{2n+1}$ and $u \in USC(\overline{\Omega}), \quad v \in LSC(\overline{\Omega})$. We define*

$$M_\varepsilon = \sup_{\overline{\Omega}\times\overline{\Omega}} \left(u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} \right)$$

with $\varepsilon > 0$. Let us suppose there exists $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$, such that:

$$\lim_{\varepsilon\to 0} \left(M_\varepsilon - (u(x_\varepsilon) - v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}) \right) = 0$$

Then we have:

$$i) \quad \lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0$$

$$ii) \quad \lim_{\varepsilon\to 0} M_\varepsilon = u(\widehat{x}) - v(\widehat{x}) = \sup_{\overline{\Omega}}(u(x) - v(x))$$

where $\widehat{x}$ is the limit of x_ε (up to subsequences) as $\varepsilon \to 0$.

Lemma 5.2. *Let $\Sigma_i \subseteq \mathbb{R}^{n_i}$ be a locally compact set and $u_i \in USC(\Sigma_i)$, for $i = 1, \ldots, k$. We define: $\Sigma = \Sigma_1 \times \ldots \times \Sigma_k$, $w(x) = u_1(x_1) + \ldots + u_k(x_k)$, where $x = (x_1, \ldots, x_k) \in \Sigma$, and $n_1 + \ldots + n_k = 2n + 1$. Let us suppose that $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_k)$ is a local maximum for $w(x) - \varphi(x)$, where $\varphi \in C^2$ in a neighborhood of $\widehat{x}$. Then, for every $\varepsilon > 0$, there exist $\Lambda_i \in S(n_i)$ such that*

$$(D_{x_i}\varphi(\widehat{x}), \Lambda_i) \in \overline{J}^{2,+}_{\Sigma_i} u_i(\widehat{x}_i), \quad \textit{for} \quad i = 1, \ldots, k$$

and the diagonal blocks matrix $\{\Lambda_i\}$ satisfies

$$-\left(\frac{1}{\varepsilon} + ||\Phi||\right) Id \leq \begin{pmatrix} \Lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \Lambda_k \end{pmatrix} \leq \Phi + \varepsilon\Phi^2$$

with $\Phi = D^2\varphi(\widehat{x}) \in S(2n+1)$ and the norm for Φ is:

$$||\Phi|| = \sup\{|\lambda| : \lambda \quad \textit{is an eigenvalue of } \Phi\} = \sup\{|\langle \Phi\xi, \xi\rangle| : |\xi| \leq 1\}$$

We can prove then the following result:

Proposition 5.3. *(comparison principle) Let $\Omega \subseteq \mathbb{R}^{2n+1}$ be a bounded open set, and let $k : \Omega \times \mathbb{R} \to \mathbb{R}$ be a prescribed continuous function, non-negative, non-decreasing with respect to u and independent of x. Then the comparison principle for F holds, namely: if $\underline{u}$ and $\overline{u}$ are respectively viscosity sub- and supersolution of $F = 0$ in Ω, such that $\underline{u}(y) \leq \overline{u}(y)$ for all $y \in \partial\Omega$, then $\underline{u}(x) \leq \overline{u}(x)$ for every $x \in \overline{\Omega}$.*

Proof. Let us define for $m \in \mathbb{N}$, $u_m(x) = \underline{u}(x) + \dfrac{1}{m}h(x)$ with $h(x) = g\Big(\frac{|x|^2}{2}\Big)$ where $g \in C^2$ and $g', g'' > 0$. We have

$$Dh(x) = g'x, \qquad D^2h(x) = g''x \otimes x + g'Id$$

and

$$tra(A(Dh)\ D^2h) \geq g' \inf_{p \in \mathbb{R}^{2n+1}} (tra(A(p)) = g' > 0$$

Moreover we choose g in such a way that $\|h\|_\infty < +\infty$. Our aim is to show that

$$\sup_\Omega (u_m - \overline{u}) \leq \frac{1}{m}\|h\|_\infty$$

We suppose by contradiction that for all m large enough we have

$$M_m = \max_{\overline{\Omega}} (u_m - \overline{u}) > \frac{1}{m}\|h\|_\infty$$

Since we have $\underline{u}(y) \leq \overline{u}(y)$ for all $y \in \partial\Omega$, such a maximum is achieved at an interior point $\tilde{x}$ (depending on m). For all $\varepsilon > 0$ let us consider the auxiliary function $w_\varepsilon(x,y) = u_m(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}$. Let $(x_\varepsilon, y_\varepsilon)$ be a maximum of w_ε in $\overline{\Omega}\times\overline{\Omega}$. By Lemma (5.1) we get as $\varepsilon \to 0$, up to subsequences, $x_\varepsilon, y_\varepsilon \to \tilde{x} \in \overline{\Omega}$, $\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = o(1)$, $u_m(x_\varepsilon) - \overline{u}(y_\varepsilon) \to u_m(\tilde{x}) - \overline{u}(\tilde{x}) = M_m$ with $u_m(x_\varepsilon) \to u_m(\tilde{x})$ and $\overline{u}(y_\varepsilon) \to \overline{u}(\tilde{x})$. We can suppose without restriction that $\tilde{x} \neq 0$. Since $\tilde{x}$ is necessarily in Ω, for ε small enough we have $x_\varepsilon, y_\varepsilon \in \Omega$. By Lemma (5.2), there exist $X, Y \in S(n)$ such that, if $p_\varepsilon := \frac{(x_\varepsilon - y_\varepsilon)}{\varepsilon}$, we have

$$(p_\varepsilon, X) \in J^{2,+}u_m(x_\varepsilon), \qquad (p_\varepsilon, Y) \in J^{2,-}\overline{u}(y_\varepsilon),$$

$$-\frac{3}{\varepsilon}Id \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{12}$$

Moreover u_m is a strictly viscosity subsolution of

$$F(x, u_m - \frac{1}{m}h(x), Du_m - \frac{1}{m}Dh(x), D^2u_m) = -\frac{g'}{m}\frac{1}{f(Du_m - \frac{1}{m}Dh(x))}$$

where $f(p) = (1+|p|^2)^{\frac{3}{2}}$. Therefore by denoting $p_\varepsilon^m = p_\varepsilon - \frac{1}{m}Dh(x)$ we have

$$\frac{g'}{m}f(p_\varepsilon^m) \leq f(p_\varepsilon)F(y_\varepsilon, \overline{u}, p_\varepsilon, Y) - f(p_\varepsilon^m)F(x_\varepsilon, \underline{u}, p_\varepsilon^m, X) =$$

$$= tra(A(p_\varepsilon^m)X) - tra(A(p_\varepsilon)Y) + f(p_\varepsilon)k(\overline{u}(y_\varepsilon)) - f(p_\varepsilon^m)k(\underline{u}(x_\varepsilon))$$

Then by using (12) we have

$$\frac{g'}{m} \leq tra(\sigma(p_\varepsilon^m)X\sigma(p_\varepsilon^m)^T - \sigma(p_\varepsilon)Y\sigma(p_\varepsilon)^T) + f(p_\varepsilon)k(\overline{u}(y_\varepsilon)) - f(p_\varepsilon^m)k(\underline{u}(x_\varepsilon)) \leq$$

$$\leq \frac{3}{\varepsilon}\Big(\sigma(p_\varepsilon^m) - \sigma(p_\varepsilon)\Big)\Big(\sigma(p_\varepsilon^m) - \sigma(p_\varepsilon)\Big)^T + f(p_\varepsilon)k(\overline{u}(y_\varepsilon)) - f(p_\varepsilon^m)k(\underline{u}(x_\varepsilon)) \leq$$

$$\leq \frac{3L_\sigma^2}{\varepsilon m^2}(g')^2|x_\varepsilon|^2 + f(p_\varepsilon)k(\overline{u}(y_\varepsilon)) - f(p_\varepsilon^m)k(\underline{u}(x_\varepsilon))$$

Now we note that $f(p_\varepsilon^m) \approx f(p_\varepsilon)$ as $m \to \infty$, and by hypotheses on k and Lemma (5.1) we get $k(\overline{u}(\widetilde{x}) - k(\underline{u}(\widetilde{x})) \leq 0$, as $\varepsilon \to 0$. Therefore by choosing $m = \varepsilon^{-2}$ and taking the limit as $\varepsilon \to 0$ we obtain a contradiction. $\square$

Proof. (of Theorem 1.1) Since we have comparison principle for both cases, by the Perron type theorem in [9] (Proposition II.1), we have that if there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ for (DP) such that $\underline{u} = \overline{u} = \varphi$ on $\partial\Omega$, then there exists a unique viscosity solution for (DP). Therefore we are interested in finding explicit sub- and supersolutions for (DP). Let $\rho \in C^2$ be a defining function for Ω. Let $V_0 = \{x \in \mathbb{R}^{2n+1} : -\gamma_0 < \rho(x) < 0\}$, $\gamma_0 > 0$ such that for every $0 \leq \gamma \leq \gamma_0$ the cylinder Ω_c^γ still satisfies (1), where $\Omega^\gamma = \{x \in \mathbb{R}^{2n+1} : \rho(x) < -\gamma\}$. Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a sequence of smooth functions uniformly convergent to φ on $\partial\Omega$; let finally $\widetilde{\varphi}_\varepsilon$ be a smooth extension of φ_ε on Ω. We define $\underline{u}_\varepsilon(x) = \widetilde{\varphi}_\varepsilon(x) + \lambda\rho(x)$ and $\overline{u}_\varepsilon(x) = \widetilde{\varphi}_\varepsilon(x) - \lambda\rho(x)$, with $\lambda > 0$. It holds $\underline{u}_\varepsilon = \overline{u}_\varepsilon = \varphi_\varepsilon$ on $\partial\Omega$ and for λ large enough we have $\underline{u}_\varepsilon \leq \overline{u}_\varepsilon$ on V_0. Now by (1), for every $x \in V_0$, one has:

$$\lim_{\lambda\to\infty} -tra(\widetilde{A}(D\underline{u}_\varepsilon)D^2\underline{u}_\varepsilon) + k(x, \underline{u}_\varepsilon) = -\mathcal{C}_x^{\Omega_c^\gamma} + k(x, s) \leq 0;$$

$$\lim_{\lambda\to\infty} -tra(\widetilde{A}(D\overline{u}_\varepsilon)D^2\overline{u}_\varepsilon) + k(x, \overline{u}_\varepsilon) = \mathcal{C}_x^{\Omega_c^\gamma} + k(x, s) \geq 0.$$

Let x_0 be the center of the smallest ball $B(x_0, R)$ containing Ω and let us introduce the function $h(x) = -\sqrt{R^2 - |x|^2}$, so that $tra(\widetilde{A}(Dh)D^2h) = 1/R$. We define

$$\underline{v}_\varepsilon = \begin{cases} \underline{u}_\varepsilon(x) & \forall x \in V_0 \\ h(x) - M_1 & \forall x \in \Omega \setminus V_0 \end{cases}, \quad \overline{v}_\varepsilon = \begin{cases} \overline{u}_\varepsilon(x) & \forall x \in V_0 \\ M_2 & \forall x \in \Omega \setminus V_0 \end{cases}$$

with $M_1 \geq \sup_{V_0}(h(x) - \underline{u}_\varepsilon)$ and $M_2 \geq \sup_{V_0} \overline{u}_\varepsilon$. Therefore $\underline{v}_\varepsilon$ and $\overline{v}_\varepsilon$ are respectively sub- and supersolution of (DP) with boundary datum φ_ε. Then there exists a unique viscosity solution of (DP). From comparison principle

$$\sup_\Omega |u_\varepsilon - u_{\varepsilon'}| = \sup_{\partial\Omega} |u_\varepsilon - u_{\varepsilon'}| = \sup_{\partial\Omega} |\varphi_\varepsilon - \varphi_{\varepsilon'}|$$

Since viscosity solutions are stable with respect to uniform convergence (see [3]) then u_ε uniformly converges to the unique solution of (DP). □

6 Lipschitz viscosity solutions

In this section we are looking for Lipschitz continuous viscosity solutions of (DP). We will regularize in elliptic way our operator in order to obtain a smooth solution u_ε; then we will prove a uniform gradient estimate for Du_ε by using a Bernstein method and finally we will get our solution by taking

the uniform limit of u_ε. For $0 < \varepsilon \leq 1$, let us set $A^\varepsilon(p) := A(p) + \varepsilon Id$; then A^ε is strictly positive definite and

$$F^\varepsilon(x, u, Du, D^2u) := -tra(\widetilde{A}^\varepsilon(Du)D^2u) + k(x, u)$$

is elliptic. We consider the following perturbed Dirichlet Problem:

$$\begin{cases} F^\varepsilon(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u(x) = \varphi(x), & \text{on } \partial\Omega. \end{cases} \qquad (DP_\varepsilon)$$

We prove:

Proposition 6.1. *Let $k \in C^1(\overline{\Omega} \times \mathbb{R})$ and $\varphi \in C^{2,\alpha}(\partial\Omega)$, $0 < \alpha < 1$. If (4) and (5) hold, then (DP_ε) admits a solution $u^\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ such that*

$$\max_{\overline{\Omega}} |Du^\varepsilon| = \max_{\partial\Omega} |Du^\varepsilon| \qquad (13)$$

Proof. The first statement is a consequence of the ellipticity of F^ε (see [5]). Now let $\widetilde{A}^\varepsilon = \{\widetilde{a}^\varepsilon_{ij}\}$, therefore we can write

$$-\sum_{i,j=1}^{2n+1} \widetilde{a}^\varepsilon_{ij}(Du^\varepsilon)\partial_{ij}u^\varepsilon + k(x, u^\varepsilon) = 0 \qquad (14)$$

By differentiating (14) with respect to x_k, we get:

$$-\sum_{i,j=1}^{2n+1}\Big(\sum_{l=1}^{2n+1} \frac{\partial \widetilde{a}^\varepsilon_{ij}}{\partial_l u^\varepsilon}\partial_{lk}u^\varepsilon\Big)\partial_{ij}u^\varepsilon - \sum_{i,j=1}^{2n+1} \widetilde{a}^\varepsilon_{ij}\partial_{ijk}u^\varepsilon + \frac{\partial k}{\partial x_k} + \frac{\partial k}{\partial u^\varepsilon}\partial_k u^\varepsilon = 0$$

We multiply by $\partial_k u^\varepsilon$ and we take the sum over k:

$$-\sum_{i,j,l,k=1}^{2n+1} \frac{\partial \widetilde{a}^\varepsilon_{ij}}{\partial_l u^\varepsilon}\partial_{kl}u^\varepsilon\partial_{ij}u^\varepsilon\partial_k u^\varepsilon - \sum_{i,j,k=1}^{2n+1} \widetilde{a}^\varepsilon_{ij}\partial_{ijk}u^\varepsilon\partial_k u^\varepsilon + \sum_{k=1}^{2n+1} \frac{\partial k}{\partial x_k}\partial_k u^\varepsilon + \frac{\partial k}{\partial u^\varepsilon}|Du^\varepsilon|^2 = 0 \qquad (15)$$

We set now $v^\varepsilon = |Du^\varepsilon|^2 = \sum_{k=1}^{2n+1} \partial_k u^\varepsilon$, so that

$$\partial_i v^\varepsilon = 2\sum_{k=1}^{2n+1} \partial_k u^\varepsilon \partial_{ik}u^\varepsilon, \qquad \partial_{ij}v^\varepsilon = 2\sum_{k=1}^{2n+1}(\partial_{jk}u^\varepsilon\partial_{ik}u^\varepsilon + \partial_k u^\varepsilon\partial_{ijk}u^\varepsilon)$$

By substituting in (15), we have

$$-\sum_{i,j,l=1}^{2n+1} \frac{1}{2}\frac{\partial \widetilde{a}^\varepsilon_{ij}}{\partial_l u^\varepsilon}\partial_{ij}u^\varepsilon\partial_l v^\varepsilon - \sum_{i,j=1}^{2n+1} \frac{1}{2}\widetilde{a}^\varepsilon_{ij}\partial_{ij}v^\varepsilon +$$

$$+\sum_{i,j,k=1}^{2n+1}\widetilde{a}_{ij}^{\varepsilon}\partial_{jk}u^{\varepsilon}\partial_{ik}u^{\varepsilon}+\sum_{k=1}^{2n+1}\frac{\partial k}{\partial x_k}\partial_k u^{\varepsilon}+\frac{\partial k}{\partial u^{\varepsilon}}v^{\varepsilon}=0 \tag{16}$$

By Schwarz theorem and by (14), it holds:

$$\sum_{i,j,k=1}^{2n+1}\widetilde{a}_{ij}^{\varepsilon}\partial_{jk}u^{\varepsilon}\partial_{ik}u^{\varepsilon}\geq\frac{(\sum_{i,j=1}^{2n+1}\widetilde{a}_{ij}^{\varepsilon}\partial_{ij}u^{\varepsilon})^2}{tr\widetilde{A}^{\varepsilon}}\geq(1+v^{\varepsilon})^{\frac{1}{2}}k^2$$

Therefore by using (16) and hypothesis (5) we get

$$\sum_{i,j=1}^{2n+1}\frac{1}{2}\widetilde{a}_{ij}^{\varepsilon}\partial_{ij}v^{\varepsilon}+\sum_{i,j,l=1}^{2n+1}\frac{1}{2}\frac{\partial\widetilde{a}_{ij}^{\varepsilon}}{\partial_l u^{\varepsilon}}\partial_{ij}u^{\varepsilon}\partial_l v^{\varepsilon}-\frac{\partial k}{\partial u^{\varepsilon}}v^{\varepsilon}=$$

$$=\sum_{i,j,k=1}^{2n+1}\widetilde{a}_{ij}^{\varepsilon}\partial_{jk}u^{\varepsilon}\partial_{ik}u^{\varepsilon}+\sum_{k=1}^{2n+1}\frac{\partial k}{\partial x_k}\partial_k u^{\varepsilon}\geq$$

$$\geq(1+v^{\varepsilon})^{\frac{1}{2}}k^2-\sum_{k=1}^{2n+1}\left|\frac{\partial k}{\partial x_k}\right|v^{\varepsilon\frac{1}{2}}\geq v^{\varepsilon\frac{1}{2}}\left(k^2-\sum_{k=1}^{2n+1}\left|\frac{\partial k}{\partial x_k}\right|\right)\geq 0$$

We can apply the classic maximum principle for elliptic operators (see [5]) and we obtain that $\max_{\overline{\Omega}}|v^{\varepsilon}|=\max_{\partial\Omega}|v^{\varepsilon}|$; therefore (13) holds. □

Now we write $Du^{\varepsilon}=(Du^{\varepsilon})^{\tau}+(Du^{\varepsilon})^{\nu}$, where $(Du^{\varepsilon})^{\tau}$ and $(Du^{\varepsilon})^{\nu}$ are respectively the tangential and normal component of Du^{ε} with respect to $\partial\Omega$: we need to estimate $(Du^{\varepsilon})^{\nu}=\langle Du^{\varepsilon},\nu\rangle=\frac{\partial u^{\varepsilon}}{\partial\nu}$, where ν is the outer normal to $\partial\Omega$. Then we have:

Proposition 6.2. *Let $u^{\varepsilon}\in C^{2,\alpha}(\overline{\Omega})$ be a solution of (DP_{ε}). If (3) holds then there exists C_0 depending on $|u^{\varepsilon}|, D\varphi, D^2\varphi$, such that:*

$$\sup_{\partial\Omega}\left|\frac{\partial u^{\varepsilon}}{\partial\nu}\right|\leq C_0 \tag{17}$$

Proof. Let $\rho\in C^{2,\alpha}$ be a defining function for Ω. Let $V_0=\{x\in\mathbb{R}^{2n+1}:-\gamma_0<\rho(x)<0\}$, $\gamma_0>0$ such that for every $0\leq\gamma\leq\gamma_0$ the cylinder Ω_c^{γ} still satisfies (3) where $\Omega^{\gamma}=\{x\in\mathbb{R}^{2n+1}:\rho(x)<-\gamma\}$. Let $\widetilde{\varphi}$ be a smooth extension of φ on Ω. We define, for any $\lambda>0$:

$$\underline{u}(x)=\widetilde{\varphi}(x)+\lambda\rho(x),\qquad\overline{u}(x)=\widetilde{\varphi}(x)-\lambda\rho(x)$$

We have $\underline{u} = \overline{u} = \varphi_\varepsilon$ on $\partial\Omega$ and $\underline{u} \leq u^\varepsilon \leq \overline{u}$ on $\{\rho = -\gamma_0\}$, for

$$\lambda > \max\{\frac{1}{\gamma_0}(\max_{\overline{\Omega}} \widetilde{\varphi} + \max_{\overline{\Omega}} |u^\varepsilon|), \frac{1}{\gamma_0}(\min_{\overline{\Omega}} \widetilde{\varphi} - \max_{\overline{\Omega}} |u^\varepsilon|)\}$$

Therefore $\underline{u} \leq u^\varepsilon \leq \overline{u}$ on ∂V_0. Now by (3), since $\Delta\rho > 0$ is strictly positive in a neighborhood of $\partial\Omega$, we have for λ large:

$$-tr(\widetilde{A}^\varepsilon(D\underline{u})D^2\underline{u}) + k(x,\underline{u}) = -tr(\widetilde{A}(D\underline{u})D^2\underline{u}) + k(x,\underline{u}) - \varepsilon(\Delta\widetilde{\varphi} + \lambda\Delta\rho) \leq 0;$$

$$-tr(\widetilde{A}^\varepsilon(D\overline{u})D^2\overline{u}) + k(x,\overline{u}) = -tr(\widetilde{A}(D\overline{u})D^2\overline{u}) + k(x,\overline{u}) - \varepsilon(\Delta\widetilde{\varphi} - \lambda\Delta\rho) \geq 0.$$

From the comparison principle we obtain $\underline{u} \leq u^\varepsilon \leq \overline{u}$ on V_0; then we have

$$\frac{\partial \underline{u}}{\partial \nu} \leq \frac{\partial u^\varepsilon}{\partial \nu} \leq \frac{\partial \overline{u}}{\partial \nu}, \quad \text{on } \partial\Omega$$

□

Next we estimate u_ε on $\overline{\Omega}$:

Proposition 6.3. *Let $u^\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ be a solution of (DP_ε). If (2) holds then:*

$$\sup_{\Omega} |u^\varepsilon| \leq \sup_{\partial\Omega} |u^\varepsilon| + C_1 \tag{18}$$

Proof. Let x_0 be the center of the smallest ball $B(x_0, R)$ containing Ω and let $v(x) = \sqrt{R^2 - |x|^2}$. By direct computation ($\triangle v \leq 0$ on $\overline{\Omega}$)

$$F^\varepsilon(v) = -tr(\widetilde{A}(Dv)D^2v) + k(x,v) + \varepsilon\triangle v \leq -\frac{1}{R} + k(x,v) \leq 0;$$

$$F^\varepsilon(-v) = tr(\widetilde{A}(D(-v))D^2(-v)) + k(x,-v) - \varepsilon\triangle v \geq 0.$$

Therefore $F^\varepsilon(v) \leq F^\varepsilon(u^\varepsilon)$ and

$$\sup_{\Omega}(v - u^\varepsilon) \leq \sup_{\partial\Omega}(v - u^\varepsilon), \qquad \inf_{\Omega}(u^\varepsilon - v) \geq \inf_{\partial\Omega}(u^\varepsilon - v)$$

Analogously $F^\varepsilon(u^\varepsilon) \leq F^\varepsilon(-v)$ and $\sup_{\Omega}(u^\varepsilon + v) \leq \sup_{\partial\Omega}(u^\varepsilon + v)$. As we have $v \geq 0$ on $\overline{\Omega}$, we proved the desired estimate. □

Finally, by the stability of viscosity solutions with respect to uniform convergence, by putting together Propositions (6.1), (6.2), (6.3), we have proved Theorem (1.3).
Next we prove a non-existence result on balls, when the prescribed curvature is a positive constant, following the idea in [1].

Proposition 6.4. *Let $B \subseteq \mathbb{R}^{2n+1}$ be the ball with center x_0 and radius R and let us suppose that k is a positive constant. If u is a Lipschitz continuous viscosity solution of $F = 0$ in $\overline{B}$, then necessarily it holds $R \leq 1/k$.*

Proof. Let $0 \leq r \leq R$ and let us consider the function $\phi(x) = M - \sqrt{r^2 - |x - x_0|^2}$, for some constant M. We have that $\phi \in C^2(B)$ and

$$tra(\widetilde{A}(D\phi)D^2\phi) = \frac{1}{r}$$

By the Lipschitz regularity of u on $\overline{B}$ we can choose M such that $u - \phi$ has a maximum at an interior point $\bar{x} \in B$; then we get (u is a viscosity subsolution of $F = 0$ as well) $F(\bar{x}, u(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \leq 0$, that is:

$$k \leq tra(\widetilde{A}(D\phi(\bar{x}))D^2\phi(\bar{x})) = \frac{1}{r}$$

for every $0 \leq r \leq R$. This ends the proof. □

7 Some examples and counterexamples

Here we show by easy counterexamples that the Strong Comparison Principle and the Hopf Lemma do not hold for the characteristic operator $\mathcal{T}$.

Example 7.1. *Let us consider the ball $B := B(0, R) \subseteq \mathbb{R}^{2n+1}$. We define the two functions $u, v : B \to \mathbb{R}$*

$$u(x) = -\sqrt{R^2 - |x_{2n+1}|^2}, \qquad v(x) = -\sqrt{R^2 - |x|^2}$$

We have

$$\begin{cases} \mathcal{T}(u) = \mathcal{T}(v) = 1/R & \text{in } B \\ u \leq v, & \text{in } B \end{cases}$$

and $u(x) = v(x)$ for all the $x \in B$ of the form $x = (0, \ldots, 0, x_{2n+1})$.

Example 7.2. *Let us consider the two functions of the previous example. Let*

$$D = \{x \in \mathbb{R}^{2n+1},\ s.t. \quad g(x) < 0\}, \qquad g(x) = x_2^2 + \ldots + x_{2n+1}^2 - x_1.$$

We set $\Omega := B \cap D$. Let $p = (0, \ldots, 0) \in \partial\Omega$, then

$$\nu = -Dg(p) = (1, 0, \ldots, 0), \qquad Du(p) = Dv(p) = 0$$

We have

$$\begin{cases} \mathcal{T}(u) \geq \mathcal{T}(v) & \text{in } \Omega \\ u < v, & \text{in } \overline{\Omega} \setminus \{p\}, \quad p \in \partial\Omega \\ u(p) = v(p) & p \in \partial\Omega \end{cases}$$

and $\dfrac{\partial u}{\partial \nu}(p) = \dfrac{\partial v}{\partial \nu}(p) = 0.$

Next we give some explicit examples of domains with the related characteristic curvature.

Example 7.3 (characteristic curvature of the spheres)**.** *Let us consider* $H(x,y) = (1/2)(|x|^2 + |y|^2)$ *as Hamiltonian function; for any positive constant* E *the isoenergetic surface of* H *is a sphere* S_R^{2n+1} *of radius* $R = \sqrt{2E}$ *and we have* $\mathcal{C}^{S_R^{2n+1}} = 1/R$*.*

Example 7.4 (characteristic curvature of cylinder type domains - 1)**.** *Let us consider* $H(x,y) = (1/2)|x|^2$ *as Hamiltonian function in* $\mathbb{R}^2 \times \mathbb{R}^2$*; for any positive constant* E *the isoenergetic surface of* H *is a cylinder* $C_1 = S_R^1 \times \mathbb{R}^2$ *with circles of radius* $R = \sqrt{2E}$ *and we have* $\mathcal{C}^{C_1} = 0$*.*

Example 7.5 (characteristic curvature of cylinder type domains - 2)**.** *Let us consider* $H(x,y) = (1/2)(x_1^2 + y_1^2)$ *as Hamiltonian function in* $\mathbb{R}^2 \times \mathbb{R}^2$*; for any positive constant* E *the isoenergetic surface of* H *is a cylinder* $C_2 = S_R^1 \times \mathbb{R}^2$ *with circles of radius* $R = \sqrt{2E}$ *and we have* $\mathcal{C}^{C_2} = 1/R$*.*

Remark 7.6. *By the previous two examples we see that the two isometric hypersurfaces* C_1 *and* C_2 *in* $\mathbb{R}^2 \times \mathbb{R}^2$ *have different characteristic curvature: indeed the isometry that exchanges* x_2 *to* y_1 *is not a symplectic diffeomorphism.*

References

[1] E. Bedford, B. Gaveau, *Hypersurfaces with Bounded Levi Form*, Indiana University Journal, 27, No. 5 (1978), 867-873.

[2] A. Bogges, *CR Manifolds and the Tangential Cauchy-Riemann Complex*, Studies in Advanced Mathematics, 1991

[3] M.G. Crandall, H.Ishii and P.L. Lions, *User's guide to viscosity solutions of second order Partial differential equations.* Bull. Amer. Soc. 27, (1992), pp 1-67.

[4] F. Da Lio, A. Montanari, *Existence and Uniqueness of Lipschitz Continuous Graphs with Prescribed Levi Curvature*, Ann. Inst. H. Poincar Anal. Non Linaire 23 (2006), no. 1, 128.

[5] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, second edition, Springer-Verlag, 1983

[6] H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*. Birkhäuser Verlag, Basel, 1994

[7] J.G. Hounie, E. Lanconelli, *An Alexander type Theorem for Reinhardt domains of $\mathbb{C}^2$. Recent progress on some problems in several complex variables and partial differential equations*, Contemp. Math., Amer. Math. Soc., Providence, RI, 400,129–146, 2006.

[8] J.G. Hounie, E. Lanconelli, *A sphere theorem for a class of Reinhardt domains with constant Levi curvature*, Forum Mathematicum 2008 20:4 , 571-586

[9] H. Ishii, P.L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, J. Differential Equations 83 (1990), no. 1, 26–78.

[10] E. Lanconelli, A. Montanari, *Pseudoconvex Fully Nonlinear Partial Differential Operators. Strong Comparison Theorems*, J. Differential Equations 202 (2004), no. 2, 306-331

[11] V. Martino, *A symmetry result on Reinhardt domains*, Differential and Integral Equations, Volume 24, Numbers 5-6 (2011), 495-504

[12] V. Martino, A. Montanari, *Graphs with prescribed the trace of the Levi Form*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (2006), no. 2, 371382.

[13] Z. Slodkowski, G. Tomassini, *Weak solutions for the Levi equation and envelope of holomorphy,* J. Funct. Anal. 101 (1991), no. 2, 392–407.

[14] Z. Slodkowski, G. Tomassini, *The Levi equation in higher dimensions and relationships to the envelope of holomorphy,* Amer. J. Math. 116 (1994), no. 2, 479–499.